Reputation Effects in Repeated Games with Learning

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Contents

Abstract

I introduce heterogeneous geodesic learning, a signal communication model based on shortest-path distance on an underlying graph among short-run players of entrants. A long-run incumbent tries to establish a reputation of a commitment type and maximize its long-run payoff by playing an entry deterrence repeated game, each period playing with one myopic entrant. This seemingly intractable geodesic belief updating rule is shown to satisfy 5 core properties outlined in Cripps (2018) [5] and hence can be mapped into a Bayesian updating as shown in proposition 1 of the same paper. Next we utilize theorem by Watson (1993) [22] on geodesic learning framework to argue that a reputation can be established in finite number of periods without equilibrium, with only best-responding agents. Depending on my framework parameters as well as edge distribution of the graph, reputation can be established faster or slower, resulting in higher or lower maximized payoff. Simulation results are consistent with analytical results, showing an upper bound of 94.6% of the total Stackelberg payoff achieved by a perfectly patient long-run player. Market entry simulations for various specifications were also completed.

to my family

1 Introduction

Reputation effects arise in games where there is a discrepancy between agents and their goals within a specific time frame. A long-run player can have the aim to establish an "identity", to which its opponents will attach a probability and take into account the possibility of this "identity" in their actions. This identity can be useful to the long-run agent in terms of future payoffs. Entry deterrence games are of particular interest in modeling reputation effects in repeated games. If the long-run incumbent is patient enough, it can be reasonable to follow a non-equilibrium strategy in the beginning stages of the repeated game and disincentivize short-run players from entering the market. In other words, a weak incumbent may be motivated to mimic a crazy type and establish a reputation of an aggressive market monopolist.

The conventional Chainstore game assumes that the agents are endowed with perfect monitoring capabilities and that they can perfectly observe the outcomes of the previous games. This case leads to the Chainstore paradox: with subgame perfect Nash equilibrium all agents should enter the game, because in the last stage the incumbent has no incentive to fight and recursively in no stage the incumbent will adopt an aggressive strategy. However, this is inconsistent with empirical evidence and the puzzle is solved by introducing the notion of perfect Bayesian Nash equilibrium. Such equilibrium is reached by endowing the entrants with beliefs regarding the commitment and normal types of the incumbent. Having obtained perfect observations of previous plays, the incumbent (both sane and insane) successfully establishes the reputation of a crazy player and deters entry into almost all stages of the game and achieves long-run Stackelberg payoff under certain conditions on the discount factor.

I introduce a model with imperfect observations of the previous play with learning: the mechanism of learning is that of a communication diffusion resting on an underlying graph network of entrants. I will model an entry deterrence Chainstore repeated game with a twist of imperfect monitoring and heterogeneous belief updating to account for spatial disparities of the network. Attaching network-specific beliefs will allow me to introduce heterogeneity in myopic entrants' belief conjectures. Note that several models have attempted to account for imperfect monitoring and neighbor-based play history observations while relating these to the future discount factor δ needed to achieve the Stackelberg payoff (Raub and Weesie (1990) [18]).

The properties of communication or signal diffusion, as I will specify in the next sections, stems from the core properties of the network. The edges of graphs can otherwise be referred to as communication channels. If edges are comparatively more than the nodes, the signals will take less time (edges) to travel to other nodes, which enhances the learning process. Thus, one can conjecture that on dense networks reputation effects are stronger and entries are more sparsely distributed in time. The properties and convergence to equilibrium will also depend on the specific network structure. A key differentiating feature in the proposed model is the heterogeneity of beliefs. A logical way to understand the heterogeneity in transferring information is using the length of shortest path between nodes as a proxy for understanding how much of the communication is lost. Nodes that are connected with one edge will have perfect communication, while nodes connected minimally by length k path will have the communication discounted by some $f(k)$. In a sense, not only are future payoffs discounted but also the communications. This discount factor itself depends on the specific distribution of the edges and nodes. For example, Raub and Weeesie (1990) [18] show how the time discount factor can be related to the probabilistic framework of the network formation, such as the probability of a node existing between two edges. This probability with the observable set of histories determines the PBE of the game. In this case, similar to the argument of the probabilistic framework of edge formation, one can also analyze the discount factor of communication or learning and relate it to the equilibrium of the repeated game. Shifting the perspective from the time horizon to the space dimension enables me to draw the properties of the learning rule from the characteristics of the underlying graph. Thus, my model builds up on the Kreps and Wilson model [16] by adding another dimension of learning: spatial learning.

Another puzzle surrounding this project is concerned with why the agent should be induced or interested in communicating its play's outcome. As specified, the monopolist is not going to play with the same entrant anymore, so the entrant has no incentive in clearly communicating the monopolist's type to later players. However, I will abstract from this complexity and will leave this for a possible extension section. There are several ways in which a researcher can explain this, such as the outcome payoff being shared between the agents or the implementation of utility transfer models; however these are beyond the scope of this thesis.

The model proposed in this thesis will first characterize a geodesic belief updating rule, then prove that it is Bayesian or can be mapped to a Bayesian updating. Next, it will state a reputation result based on the model, analyze market entry frequencies based on graph characteristics, as well as several growth results on the number of nodes. The upcoming sections of this thesis are structured as follows: the literature review will go on by emphasizing key features of seminal papers on the topic of reputation effects and delineate their setbacks. Empirical evidence from economics paper citations will be presented in the next section, showing that collaborations with more dense networks have a higher citation index. I will then present the model setup of the heterogeneous geodesic learning. These sections will be followed by the introduction of 6 critical properties on this learning or belief updating rule, after which I will refer back to 2 existing results to show that the simulations of the game will produce sensible results even with best-responding agents without the need for an equilibrium. Finally, I will represent simulation results, characterize market entry with comparative statics of the underlying parameters and discuss results with possible extensions.

2 Literature review

Preliminary literature starts with Fudenberg and Maskin (1983) [9] exploring strategic rivalry and extensively expanding notions around repeated play and plausible payoff sets, by introducing different specifications of Folk Theorems. The literature evolving around repeated games is multidimensional, including theoretical and empirical work in network theory, collaborative and experimental games, business innovation and technology adoption. This subfield is of particular interest, because it opens new opportunities for explaining the seemingly irrational agent behavior and payoff achievements previously implausible to do so. The literature of reputation effects in repeated games is rooted in the seminal models of Kreps and Wilson (1981) [16]. I will be using payoff matrices from this model to derive results. Fudenberg and Levine (1992) [8] suggest an imperfectly observed reputation model and characterize the lower and upper bounds on the possibly achievable payoff sets of the long-run player. In their 1989 paper [7], they also characterize the equilibrium selection and prove that a positive probability of the commitment type is enough to achieve the Stackelberg payoff pending on the discount factor. As shown in results section, my simulations have achieved an upper bound of 94.6 % from the total Stackelberg payoff of the repeated game. However, the majority of the literature abstracts from the endogenous effects of the communication between the short-run agents or assumes they observe the history of previous plays in the same way, i.e. there is a single belief value maintained throughout the game and all agents make decisions based on this value. The belief sets are homogeneous across the entrants. There is extensive literature separately on the collaboration games and games on networks, which characterize equilibria and collaboration conditions in general networks. Logically the underlying network has to affect the collaboration tendencies in, for example, repeated prisoner's dilemma games, hinting towards more collaboration on denser graphs, argued in Coleman JS (1988) [4]. Dall'Asta, Marsili and Pin (2012) [6] go on to define a new concept of collaborative equilibria in trees and random graphs, which establish a Nash equilibrium of repeated public good contribution games with punishment strategies on networks (which can be mapped to repeated prisoner's dilemma games). Significant contribution in the best-response dynamics on networks and the speed of signal transmission (or learning) is made in paper series by Golub and Jackson (2011) [10], which take a step to analyze slower belief convergence according to what they define as degree-weighted homophily, as compared to shortest distance approach we will adopt, which is affected by network density but not from homophily. One can argue that the underlying graph is itself the representation of the existing homophily of agents, especially in the economic framework of market competition. Connections and communication patterns on general graphs are heavily analyzed in Goyal (2007) [12] concentrating on coordination games. Communication patterns in networks are introduced in Lippert and Spagnolo (2009) [17] leading to different equilibrium restrictions based on specific graph properties. An interesting approach to intertwine graph theory, combinatorics and signaling games is introduced in Hu, Skyrms, Tarres (2018) [13], which uses Roth-Erev reinforcement learning rule to create a correspondence between signals and maximize the expected payoff of the game. In most of the cases, network literature concentrates on repeated coordination, public good provision and seller-buyer games, mostly characterizing graph properties necessary for Nash equilibrium, characterizing payoff boundaries as the discount factor converges to 1, as well as exploring strategies and equilibria on specific graphs, such as star-shaped, linear and circular ones. Golub and Jackson (2010) [14] also introduce a heterogeneous updating rule similar to that of DeGroot learning to understand restrictions required for "convergence to truth" states. Their and Sadler's literature mostly analyzes belief influences, conformity and convergence properties, while the reputation literature predominantly bases on Kreps and Wilson's model of homogeneous beliefs with exogenous information and a Perfect Bayesian Nash equilibrium concept of a well-chosen belief updating rule. The latter is appropriate for getting analytically tractable and sensible results; however, the belief rule can be perturbed in a way to give more room for possible research extensions, although risking to not find a tractable equilibrium concept. The proposed model of heterogeneous belief updating, proposed in this research project, falls into this category and a comprehensive scanning of belief updating literature will also be utilized. The closest existing framework to our model is given by Raub and Weesie (1990) [18]. They analyze several specifications of previous play history-based belief updating in a repeated prisoner's dilemma with a probabilistic graph framework and show that the time discount factor can be related to the probabilistic framework of the network formation, such as the probability of a node existing between two edges. These specifications include atomized (other players do not observe anyone's history), perfectly embedded (agents observe their neighbor's past play outcomes) and imperfectly embedded (neighbor's play is observed with a time lag) observation cases. The first case corresponds to the model of this thesis with no edges, the second case - to the proposed model with a perfectly connected graph and the last case - imperfectly connected graph. The authors characterize the conditions on discount factor needed to maximize the agents' payoff and show that trigger strategies are sustainable equilibria given certain conditions on the expected discount value and defection temptation propensity (depends on payoffs) with restrictions being the toughest for the atomized case. These restrictions are similar to results of my model regarding the parameter value of signal spread α as shown in my results.

My paper also finds motivation from Battigalli and Watson (1997) [1] on establishing reputation effects with heterogeneous beliefs. Notable findings on network based belief updating are summarized by Jackson [15], stating significant results on DeGroot learning. The belief updating rule I will propose is more generalized than that of the DeGroot learning, which only utilizes neighbor's beliefs with influence scores to update beliefs for the next stage. My model, while using the idea of the DeGroot learning, proposes a rule which enables all players in the network to influence the belief of a single player, independent of whether they are neighbors. The key point is the existence of a path between the agents and the influencer. The influence score (or weight) as defined in DeGroot will be a function of the shortest path distance, as specified in the next sections. One of the fundamental existing results in DeGroot modeling, is that a consensus can be reached on graphs only if the graph is connected and aperiodic, i. e. the greatest common divisor of all cycle lengths is 0 [15]. Introducing imperfect communication between agents in the network after an outcome of play was observed is motivated by the heterogeneity of accessing information and connectivity disparities in the social network. Thus, we will introduce a fixed network of agents that will communicate to each other the outcomes of their play. The comparison of speed of play outcome communication and hence reputation establishment between different graphs can be related to the stochastic dominance of its edge distributions argued in section 8.

As shown in Cripps (2018) [5] any updating rule can be characterized by a homeomorphism to a Bayesian learning rule under certain conditions. This is the key for mixing Kreps and Wilson's general framework of reputation with Watson's results and a more reasonable network-based belief learning rule that is intractable. *Fixing what I define as geodesic learning, which depends on graph properties, I will use Cripps' argument to show that this learning rule can be mapped into a Bayesian learning framework, which will enable me to use Watson's reputation refinement result without equilibrium (*1993*) [22]*. Hence, I will not introduce any equilibrium concept other than rationalizability and best-response dynamics. As experimental evidence in cooperative games suggest (Dal Bó et al (2011) [2]) cooperation is decreased with experience if the outcome is not supported by the equilibrium; however, this does not pose a threat to our model robustness, because the best response dynamics ensures that the agents play Nash equilibrium in each period game (and hence the outcome is supported by equilibrium), even though the total game equilibrium might be different as shown in Kreps and Wilson's model. The latter argues that the game will be classified into 3 main phases: the long-run agents play the pure commitment strategy in the beginning of the game while the short run agents stay out, then both sides start to play mixed strategies and finally the long-run agent accommodates and the entrant enters.

Empirical entry threat analyses have also been performed, such as Goolsbee and Syverson (2004) [11], but are mainly focusing on price mechanisms and abstract from signal transmission and theoretical learning rules. Reputation effects have also been empirically analyzed in patent citation networks, a simple novel example of which I will introduce in the empirical evidence section.

The intersection of repeated coordination game theory, network and communication theory and entry deterrence games has limited coverage in the existing literature of reputation effects. At least one of the above-mentioned components is missing in all papers I have encountered: some model reputation effects in entry deterrence games without an underlying heterogeneous signal transmission or complex belief updating. Others analyze games on networks heavily from the perspective of coordination games and not of strategic competition. The discerning features in all papers are the information mechanism and the choice of equilibrium concept. My paper uses geodesic learning information mechanism and a best-response dynamic equilibrium concept.

3 Empirical Evidence

I will use this section to argue that reputation effects are established faster in more connected graphs. To find an applicable setting, I have collected a database of published paper collaborations and citation data. Several specifications of citation and collaboration data were tested. I used the Scopus academic paper search engine provided by Elsevier to collect data on papers in the field of Economics, Econometrics and Finance starting from the publishing year of 1860. $1,257,444$ articles' details were gathered. Each data cell included information about the paper title, year, authors and their identification numbers in Elsevier, publishing source, starting and ending pages of the article and citation counts. For each author ID, I calculated the average number of collaborators for each year t based on published papers before year t. Then for each paper published at year t, the average number of collaborators for its authors at year t was summed. I denoted this as the connectivity index of the paper. The average number of collaborators at t is a proxy of the average number of edges a specific author ID has managed to establish before year t in the network of authors. Summarizing this "collaboration inclination" indices for each paper serves as a proxy for the paper's position in the network of authors, showing how well connected the paper is in the network and its capability to establish a reputation through the collaboration channels of its author(s). We conjecture that papers that have more channels to spread their reputation have higher citation numbers. We also include the number of authors as an explanatory variable to test this hypothesis to account for upward bias in the summed connectivity index, even though the correlation coefficient for author count and citation count was found to be 0.01. Because computing these indices was computationally costly, I have tested the hypothesis with two specifications. For the first specification, papers in $2000 - 2004$ were tested. Only papers that had at least 1 citation were included and the number of papers included from each year was capped at either 20000 or the earliest paper with minimum 1 citation. For the second specification, papers from $2000 - 2009$ were tested with top 500 cited papers included from each year. The patterns of the variables are shown in Figure 1.

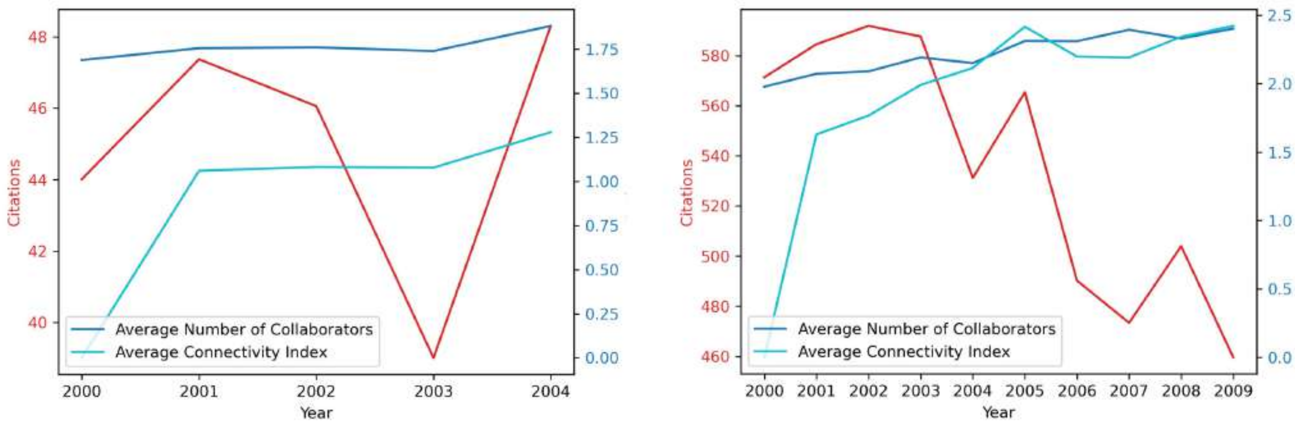

Figure 1: Explanatory variables, time series for specification 1 (left) and 2 (right)

Yearly and publisher/publishing source fixed effects were included. No publisher was found to significantly impact the number of citations. Summary of results is presented in Table 1. Specification 1 was tested only with yearly fixed effects, while specification 2 was tested with yearly and both yearly and publisher fixed effects. For the first specification both regressions resulted in highly significant paper connectivity index, while for specification 2 the strongest impact was noted when both fixed effects were included. The latter also had the highest R-squared value. Ramsey RESET test for omitted variables was performed for the last regression and an F-value of 5.92 concluded that the model has no omitted variables.

Table 2: Empirical evidence results

	Specification 1		Specification 2			
	no fe	yr fe	no fe	yr fe	source fe	both
auth	10.91^{***}	10.75^{***}	-5.303	-3.267	-1.349	0.822
	(0.526)	(0.527)	(5.698)	(5.688)	(6.067)	(6.057)
conn	4.166^{***}	4.445^{***}	4.510^{*}	6.174^{**}	4.033	6.133^{**}
	(0.283)	(0.291)	(2.465)	(2.511)	(2.511)	(2.564)
const	21.54^{***}	25.86^{***}	539.1^{***}	577.8^{***}	423.7	531.4
	(1.005)	(1.436)	(13.81)	(24.82)	(463.7)	(462.8)
R-squared	0.011	0.012	0.001	0.011	0.212	0.220
obs	81,610	81,610	5,000	5,000	5,000	5,000

Notes: $^{*}\,p < 0.1$; $^{**}\,p < 0.05$; $^{***}\,p < 0.01$

4 Model Setup

A long-run player faces a sequence of short-run entrants, one at a time. Each period a finite entry-deterrence game is played, the outcome of which is observed heterogeneously. Compared to the existing models, our proposed framework enables for richer interpretation by allowing the entrants to observe outcomes via signals of their short-run fellow entrants differently based on their connectivity. Denote one-period game as $G = \{\Delta A_1, \Delta A_2; u_1, u_2\}$, where $A_1 \in \{\text{Fight, Accommodate}\}$ is the action space of the incumbent and $A_2 \in \{In, Out\}$ is the action space of short-run entrants and $\Delta A_1, \Delta A_2$ are mixed strategies. $u_i : \Delta A_1 \times \Delta A_2 \longrightarrow \mathbb{R}$ is the payoff function. The Bayesian incumbent has two types - normal and the commitment type. Commitment type always fights (which is the Stackelberg payoff for this type). Payoff matrices are shown below for commitment (left) and sane (right) types. I have borrowed these payoffs from the conventional Kreps and Wilson's model. Abstracting from the notation used later, let us assume that the short-run agent believes that the incumbent's type is crazy with probability p. Then, $u_2(In) = p(b-1) + (1-p)b$ and $u_2(Out) = 0$. Entering will be best-response

		Entrant	
		In	Out
Incumbent	$Fight$	$(0, b-1)$	$(a, 0)$
	Acc	$(-1, b)$	$(a, 0)$

		Entrant	
		In	Out
Incumbent	$Fight$	$(-1, b-1)$	$(a, 0)$
	Acc	$(0, b)$	$(a, 0)$

if and only if $u_2(In) > u_2(Out) \implies p(b-1) + (1-p)b > 0 \implies b - p > 0 \implies p < b$.

n short-run players (entrants) are connected on a graph/network. With some abuse of notation, denote $G = (n, e)$ as the underlying network, where nodes are entrants and edges are their communication channels. To define Bayesian types formally, assume the long-run player, has types from set Θ, characterized by parameter $\theta \in \{\text{crazy, sane}\} := \Theta$. Assume entrants hold initial beliefs $\mu^0 = (\mu_1^0, \mu_2^0, ..., \mu_n^0) \in \Delta(\Theta)$ at stage 0 when the game has not started, where $\Delta(\Theta)$ denotes the simplex of belief parameters. Incumbent engages in a repeated game at periods $t = 1, 2, ..., n$. Incumbent starts playing with one and only one entrant at each period. After each period t, the outcome of that period and all previous periods constitute an outcome set $\omega^t = \{\omega_j^1, \omega_k^2, ..., \omega_m^t\}$, where superscripts denote the period of outcome and the subscripts the index of the entrant engaged in the one-period game. Keeping track of the short-run players is necessary for the belief updating rule specified in the upcoming sections. Note that if some element of outcome ω^t is observed by some player i, the rest are are not necessarily observed by this player because of the updating rule we will impose.

After each period only the agents that are connected to some degree with the entrant who plays in that period will observe the outcome and will update their beliefs according to the shortest path distance to that player. Note that this is trivial in a connected graph, because all agents will observe all outcomes although will update their beliefs differently. The latter will become furthermore trivial in a perfectly connected graph where all agents will observe the outcomes and update their beliefs identically, because the shortest path length from any agent to any other one is 1. Denote the belief of player i as μ_i^t at period t, which is the belief/probability that the incumbent is the commitment type. We can also adopt alternative notation of belief being a vector and including the complementary probability of the incumbent being the sane type, i. e. $(\mu_i^t, 1 - \mu_i^t)$. Assume all entrants hold the same initial beliefs, $\mu_i^0 = \mu^0 \ \forall i$. After outcome ω_j^t at some period t with player j, beliefs are updated in the following way:

$$\mu_i^{t+1} = \mu_i^t + \alpha f(d_{ij}) \omega_j^t \quad \forall i \tag{1}$$

for all i's that are connected to j with a path, i. e. d_{ij} is defined, where d_{ij} denotes the shortest path length from i to j. With this belief updating rule we are introducing a channel of communication between short-run agents who will transfer information about the outcome of each period game. Define $\omega_j^t = 1$ if the incumbent fought in period t with entrant j, and $\omega_j^t = -1$ if she accommodated. Throughout this analysis I will adopt the first specification of belief definition, designating the probability of the crazy type. A signal of +1 will increase the belief, i. e. the agent will be more inclined towards thinking that the incumbent is the commitment type. After each period, the playing entrant j sends a signal $s_j \in \{+1, 0, -1\} = S$, where S is the signal space. Assume that there is a probability of the playing entrant to lie about the game outcome. Denote $p_{+1} := \{p_{+1}^{+1}, p_{+1}^{-1}\}$ the probabilities that the signal is $s = +1$ when the outcome of the game in that period was $+1$ and -1, respectively. Similarly, denote $p_{-1} := \{p_{-1}^{+1}, p_{-1}^{-1}\}$. Also, define $p^{\text{crazy}} = \{p_{+1}^{\text{crazy}}, p_{-1}^{\text{crazy}}\}$, the probabilities of reporting +1 and −1 respectively, when the true type of incumbent is crazy (i. e. the outcome was "Fight"). Similarly, $p^{\text{sane}} = \{p_{+1}^{\text{sane}}, p_{-1}^{\text{sane}}\}$. Throughout this analysis, we will assume that the sane type is associated with outcome −1 of accommodate. Note that $p_{-1}^{\text{crazy}} = p_{-1}^{+1}$ and $p_{+1}^{\text{sane}} = p_{+1}^{-1}$: thus, the 4 entries in each pair are identical. If we put those probabilities in a matrix, the first pair will be the rows and the second pair will be the columns. Assuming that the agents can lie about the outcome, the signals are generated from the outcome distribution $\Delta(\omega)$. The entrants have priors $\mu^0 \in \Delta(\Theta)$ and signals with probabilities $(p^\theta)_{\theta\in\Theta}$, where $\theta \in \{\text{crazy, sane}\}$. After each period an outcome of one-period game G is realized and agent i updates beliefs according to

$$\mu_i^{t+1} = \mu_i^t + \alpha f(d_{ij}) s_j^t \quad \forall i \tag{2}$$

where we have substituted ω_j^t with the signal s_j^t, a realization of either +1 or −1, $f(\cdot)$ is well-defined function on positive integers, d_{ij} is the shortest path length from i to j. The choice of function $f(\cdot)$ will reflect the properties of spatial transmission mechanism. It is intuitive to impose faster diffusion of signal among neighbors or nearest nodes on the underlying graph $G = (n, e)$. Hence, $f'(\cdot) < 0$, $f : N \times N \longrightarrow \mathbb{P}$, where $\mathbb{P}$ is the set of positive integers and N the set of short-run entrants. If the entrant stays out, it transmits a signal of 0 and beliefs are not updated. Parameter α denotes the speed of transmission. Note that beliefs need to stay in the range of $(0, 1)$; however, this concern is trivial because our results stay invariant to an affine transformation of beliefs and threshold values, so with proper normalization and tuning of $\alpha, \mu^0$and$f(\cdot)$ we can guarantee that beliefs are well-defined.

To sum up, the incumbent plays with a different entrant j in each period t. After the outcome of the game ω_j^t is realized, j sends a signal with probability distribution $(p^\theta)_{\theta\in\Theta}$ and other short-run agents who are connected to j with some path on $G = (n, e)$ observe the signal s_j^t and update their beliefs according to (2). The proposed belief updating rule (or network learning rule) satisfies several properties that will be conducive to reputation arguments in the repeated game. These properties were first popularized and comprehensively studied in Cripps (2018) [5]. The following section proceeds by showing that our seemingly intractable updating rule can be mapped onto Bayesian learning. I will call the proposed belief updating rule as geodesic learning.

5 Geodesic Learning Properties

These properties are defined and taken from Cripps (2018).

Property 5.1. *Uninformativeness: If $p^{crazy} = p^{sane}$, beliefs are not updated.*

We will ensure that this property is guaranteed apriori by $p^{\text{crazy}} \neq p^{\text{sane}}$.
Suppose entrant receives a set of signals at period t (when it is called to play and WLOG it is connected to all previous players by some path, so that t signals are reached) $\zeta = (\zeta_1, \zeta_2, ..., \zeta_t)$ and define function $m : \zeta \longrightarrow \zeta$ as a permutation of signal order.

Property 5.2. *Permutation: Belief updating rule U : signal set $\times$ prior $\longrightarrow$ posterior satisfies the permutation property if $U(\zeta, \mu^0) = U(m(\zeta), \mu^0)$, i. e. the order of received signals is irrelevant.*

Geodesic learning satisfies this property. Denote the realized signal set received by an arbitrary short-run player i at the end of period t, $s^t = (s^1_{j_1}, s^2_{j_2}, ..., s^t_{j_t})$, where j_x is the entrant index at period x. $m(s^t)$ is a permutation of s^t. Under s^t, the updated beliefs are: $\mu_i^{t+1} = \mu_i^t + \alpha \sum_{x<=t} f(d_{ij_x}) s^x_{j_x}$. Because geodesic learning accumulates all signals commutatively, under permutation the second term is not changed, given also the shortest path distance function is uniquely determined for each player j_x. Thus, for any $m(\cdot)$, $\sum_{x<=t} f(d_{im(j_x)}) m(s_{j_x}) = \sum_{x<=t} f(d_{ij_x}) s_{j_x}$.

Property 5.3. *Non-dogmatic: Belief updating rule is continuous and any belief value can be achieved through appropriate signal probabilities and outcome distributions.*

Cripps [5] argues that property 5.1 is enough to deduce property 5.3 for binary experiments (which is our case since $\|\Theta\| = 2$). Moreover,

$$\mu_i^{t+1} = \mu_i^t + \alpha \sum_{x<=t} f(d_{ij_x}) s_{j_x}^x = \mu^x \tag{3}$$

$$\alpha \sum_{x<=t} f(d_{ij_x}) s_{j_x}^x = \mu^x - \mu_i^t \tag{4}$$

$$\sum_{x<=t} f(d_{ij_x}) s_{j_x}^x = \text{Constant} \tag{5}$$

which has a solution due to choice of $f(\cdot)$.

Property 5.4. *Divisibility: If posterior beliefs are identical in cases where 1) agent updates beliefs based on each signal separately, 2) agent updates beliefs simultaneously using all available signals, then the belief updating rule is said to be divisible.*

Suppose the realized signal set received by an arbitrary short-run player i at the end of period t, $s^t = (s_{j_1}^1, s_{j_2}^2, ..., s_{j_t}^t)$ and WLOG denote some partition of it into k-sets $s_1^t, s_2^t, ..., s_k^t$, such that $s_i^t \cap s_j^t = \varnothing \quad \forall i, j$ and $\bigcup_i s_i^t = s^t$. Assume in case 1 agent receives signals separately one at a time from s^t and in case 2 receives separately k times, each time τ receiving the bunch of signals that are in the partition s_τ^t. In the former case, belief updating proceeds as follows in t steps (the number of periods):

$$\begin{aligned} \mu_i^0 &= \mu^0 \\ \mu_i^1 &= \mu^0 + \alpha s_{j_1}^1 f(d_{ij_1}) \\ \mu_i^2 &= \mu_i^1 + \alpha s_{j_2}^2 f(d_{ij_2}) \\ &... \\ \mu_i^t &= \mu_i^{t-1} + \alpha s_{j_t}^t f(d_{ij_t}) \end{aligned} \tag{6}$$

Resulting in,

$$\begin{aligned} \mu_i^t &= \mu^0 + \alpha(s_{j_1}^1 f(d_{ij_1}) + s_{j_2}^2 f(d_{ij_2}) + ... + s_{j_t}^t f(d_{ij_t})) \\ &= \mu^0 + \alpha(\sum_{l \in s^t} s_{j_l}^l f(d_{ij_l})) \end{aligned} \tag{7}$$

In the latter case of k-partitioned signals, we have:

$$\begin{aligned}
\mu_i^0 &= \mu^0 \\
\mu_i^1 &= \mu^0 + \alpha \sum_{l \in s_1^t} s_{j_l}^l f(d_{ij_l}) \\
\mu_i^2 &= \mu_i^1 + \alpha \sum_{l \in s_2^t} s_{j_l}^l f(d_{ij_l}) \\
&\dots \\
\mu_i^k &= \mu_i^{k-1} + \alpha \sum_{l \in s_k^t} s_{j_l}^l f(d_{ij_l})
\end{aligned} \tag{8}$$

Resulting in,

$$\begin{aligned}
\mu_i^t = \mu_i^k = \mu^0 &+ \alpha(\sum_{l \in s_1^t} s_{j_l}^l f(d_{ij_l}) + \dots + \sum_{l \in s_k^t} s_{j_l}^l f(d_{ij_l})) \\
&= \mu^0 + \alpha(\sum_{l \in s_1^t \cup \dots \cup s_k^t} s_{j_l}^l f(d_{ij_l})) \\
&= \mu^0 + \alpha(\sum_{l \in s^t} s_{j_l}^l f(d_{ij_l}))
\end{aligned} \tag{9}$$

where third steps using the fact that $s_i^t \cap s_j^t = \varnothing \quad \forall i, j$ and $\bigcup_i s_i^t = s^t$. The resulting posterior beliefs are the same for both cases. This is a core property of the belief updating rule that we posited and it stems from the commutative nature of signal-based updating and static distribution of the geodesic distances (shortest path lengths). It reflects the fact that agents accumulate information about incumbent linearly as new signals are generated from a given distribution, taking into consideration both compatible and conflicting signals of different periods, discounting them spatially, based on the shortest distance to the "information-spreader" giving higher credibility to those who are closer. The credibility argument is reflected in $f'(\cdot) < 0$.

Definition 5.5. *A belief updating rule is proximity-based if $f(d_{ij}) > f(d_{ik})$ if $d_{ij} < d_{ik}$ for a fixed entrant i and arbitrary j, k, such that $d_{ij} and d_{ik}$ are defined.*

Cripps [5] argues that if the belief updating rule satisfies properties 5.1-5.4, we can establish several properties on the network-based belief updating mechanism, such as the updating rule being homogeneous of degree 0 in signaling probabilities p_+, p_-. However, we are interested in characterizing our belief updating in a Bayesian way (required for later sections), for which we need the geodesic learning to satisfy the last property of the series, unbiasedness, which is where we start restricting the transmission mechanism to specific parameter values.

Property 5.6. *Unbiasedness: The updating rule $U_n : \mu^0 \times S^t \longrightarrow \mu^t$ satisfies unbiasedness property if the posterior predicted distribution equals the prior beliefs, i. e.*

$$\mu^0 \equiv \sum_{s\in S}(\sum_{\theta\in\Theta} \mu_\theta p_s^\theta)U_n^0(\mu, (p^\theta)_{\theta\in\Theta}) \tag{10}$$

Theorem 5.7. *Geodesic learning satisfies unbiasedness property for some initial belief $\mu^*(p^\theta)$.*

Proof: Because the proof proceeds by proving unbiasedness for a single signal, we will first show that this is not a concern, since a state with multiple signals can be bijectively mapped to a state with a single signal. Because the belief updating rule is divisible we can look at the agent who has accumulated evidence and needs to update when he is only called to play. Thus, at that point the agent has a collection of signals, either $+1$ or -1 and their corresponding geodesic distances. For example, suppose the available signals for some agent i are Accommodate (-1) coming from j with $d_{ij} = 2$ and Fight $(+1)$ coming from k with $d_{ik} = 4$. Hence, their total perturbance to the initial belief of i will be $d_{ij} \cdot (-1) + d_{ik} \cdot (+1) = -2+4 = 2$. Because geodesic distances and signal values are natural numbers, this is equivalent to a single signal $+1$ from some agent l with $d_{il} = 2$. Hence, due to divisibility, unbiasedness will pose a restrictive condition only on the initial belief and not on beliefs at every stage. Denote the signal probabilities as $p_{+1}^{\text{crazy}} \equiv$ probability that the incumbent fought and entrant sent signal $+1$. $p_{-1}^{\text{sane}}, p_{-1}^{\text{crazy}}, p_{+1}^{\text{sane}}$ are defined similarly. Initial belief of a random entrant i is $\mu_i^0 = \mu^0$ or $\mu_i^0 = (\mu^0, 1-\mu^0)$. When the agent receives signal $+1$ the beliefs are updated to $(\mu^0 + \alpha f(d_{ij}))$, and when the signal is -1, updated to $(\mu^0 - \alpha f(d_{ij}))$. Note that the agent cares only about the observed signal and not the true type of the incumbent. Because this learning rule (belief updating rule) is divisible by property 5.4, suppose the agent updates its beliefs just before it is called to play, i. e. when it has the most signals available observed. This corresponds to the least granular partition case discussed earlier where the partitioned set is equal to the whole set. As we will see, it will pose the least restrictions on the prior beliefs: thus, assume the agent updates only once from its original belief $\mu_i^0 \implies$ posterior predicted distribution equals

$$[\mu_i^0 p_{+1}^{\text{crazy}} + (1-\mu_i^0)p_{+1}^{\text{sane}}](\mu^0 + \alpha\sum_j f(d_{ij}) + [\mu_i^0 p_{-1}^{\text{crazy}} + (1-\mu^0)p_{-1}^{\text{sane}}](\mu_i^0 - \alpha\sum_j f(d_{ij})) \tag{11}$$

Note that because when the entrant stays out, the beliefs are not updated and agent can be thought of as not sending a signal, which means we can assume $p_{+1}^{\text{crazy}} + p_{+1}^{\text{sane}} = 1$ and $p_{-1}^{\text{crazy}} + p_{-1}^{\text{sane}} = 1$. For unbiasedness,

$$\mu_i^0 \equiv [\mu_i^0 p_{+1}^{\text{crazy}} + (1-\mu_i^0)p_{+1}^{\text{sane}}](\mu^0 + \alpha\sum_j f(d_{ij}) + [\mu_i^0 p_{-1}^{\text{crazy}} + (1-\mu^0)p_{-1}^{\text{sane}}](\mu_i^0 - \alpha\sum_j f(d_{ij}))$$

$$\mu_i^0 \equiv (\mu_i^0)^2 p_{+1}^c + \mu_i^0 p_{+1}^c \alpha\sum_j f(d_{ij}) + \mu_i^0 + \alpha\sum_j f(d_{ij}) - \mu_i^0 p_{-1}^s - p_{-1}^s\alpha\sum_j f(d_{ij}) - (\mu_i^0)^2 - \mu\alpha\sum_j f(d_{ij}) +$$

$$\mu_i^0)^2 p_{-1}^s + \mu_i^0 p_{-1}^s \alpha\sum_j f(d_{ij}) + (\mu_i^0)^2 - \mu_i^0\alpha\sum_j f(d_{ij}) -$$

$$(\mu_i^0)^2 p_{+1}^c + \mu_i^0 p_{+1}^c\alpha\sum_j f(d_{ij}) + \mu_i^0 p_{-1}^s - p_{-1}^s\alpha\sum_j f(d_{ij}) - (\mu_i^0)^2 p_{-1}^2 + \mu_i^0 p_{-1}^s\alpha\sum_j f(d_{ij})$$

Cancelling out identical terms, we get

$$\alpha\sum_j f(d_{ij}) - 2p_{-1}^s\alpha\sum_j f(d_{ij}) + 2\mu_i^0 p_{+1}^c\alpha\sum_j f(d_{ij}) - 2\mu_i^0\alpha\sum_j f(d_{ij}) + 2\mu_i^0\alpha p_{-1}^s\sum_j f(d_{ij}) \equiv 0$$

$$\alpha\sum_j f(d_{ij})[-1 + 2p_{-1}^s - 2\mu_i^0 p_{+1}^c + 2\mu_i^0 - 2\mu_i^0 p_{-1}^s] \equiv 0$$

$$2p_{-1}^s - 2\mu_i^0(1 - p_{+1}^c - p_{-1}^s) \equiv 1$$

$$\mu_i^0 \equiv \frac{2p_{-1}^s - 1}{2(p_{+1}^c + p_{-1}^s - 1)} = \mu^*$$

□

This is the condition required for prior beliefs for unbiased updating. Note that if $p_{-1}^s = p_{+1}^c$, then $\mu_i^0 = \frac{1}{2}$ no matter the exact values of p_{-1}^s and p_{+1}^c. We can be ignorant of the specific signaling probabilities given that the entrant will tell the truth with the same probability regardless of the game outcome. Also note that it is sufficient to establish the unbiasedness for μ_i^0, because the complementary probability of incumbent being sane will be satisfied automatically. The value of α does not affect the level of initial priors needed for martingale unbiasedness. False reporting probabilities are also not explicitly in this expression. Martingale unbiasedness property is the core characterizing complex beliefs and their corresponding equilibria in repeated games. Considering the comparative statics with respect to p_{-1}^s and p_{+1}^c, we have

$$\begin{aligned}\frac{\partial\mu^*}{\partial p_{-1}^s} &= \frac{2\cdot 2(p_{+1}^c + p_{-1}^s - 1) - 2\cdot(2p_{-1}^s - 1)}{4(p_{+1}^c + p_{-1}^s - 1)^2} \\ &= \frac{4p_{+1}^c + 4p_{-1}^s - 4 - 4p_{-1}^s + 2}{4(p_{+1}^c + p_{-1}^s - 1)^2} \\ &= \frac{2(2p_{+1}^c - 1)}{4(p_{+1}^c + p_{-1}^s - 1)^2}\end{aligned}$$

Thus, $\frac{\partial\mu^*}{\partial p_{-1}^s} > 0$ if $p_{+1}^c > \frac{1}{2}$ and < 0 if $p_{+1}^c < \frac{1}{2}$. The intuition behind this is the following: if the agent is inclined to tell the truth in periods of fight outcome, then the more inclined it is to also tell the truth on accommodation outcome periods, the higher the value of initial belief is required. If the agent is more likely to tell lies on fight outcomes, the opposite holds. This leads to a conjecture that the key variable is the dispersion of truthfulness based on outcome, i. e. $(p_{+1}^c - p_{-1}^s)$. Similarly,

$$\frac{\partial\mu^*}{\partial p_{+1}^c} = \frac{1 - 2p_{-1}^s}{2(p_{+1}^c + p_{-1}^s - 1)^2}$$

Comparative statics with respect to p_{+1}^c exhibits exactly opposite characteristics to that of p_{-1}^s. When $p_{-1}^s = p_{+1}^c$, both partial derivatives equal $\frac{1}{2(p_{+1}^c + p_{-1}^s - 1)}$. Using the above discussion, we can state the following theorem.

Theorem 5.8. *Given $G = (n, e)$ network of short-run entrants and a long-run incumbent involved in a repeated game, where each period agents play game $G = \{\Delta A_1, \Delta A_2; u_1, u_2\}$, if agents use geodesic belief updating rule, there is a homeomorphism F that maps these beliefs into Bayesian belief updating mechanism. Moreover, if the agents belief updating rule also satisfies property* 5.5, *then F is an identity.*

Proof: We showed that geodesic learning satisfies properties 5.1 – 5.4. Thus, using proposition 1 in Cripps (2018) [5], $\exists F : \Delta(\Theta) \longrightarrow \Delta(\Theta)$, such that

$$U(\mu, p_s) = F^{-1}[\frac{F_1(\mu)p_s^c}{\sum_{\theta\in\Theta} F_\theta(\mu)p_s^\theta}, \frac{F_2(\mu)p_s^s}{\sum_{\theta\in\Theta} F_\theta(\mu)p_s^\theta}] \tag{12}$$

where subscript s is the signal received, $F(\mu) = (F_1(\mu), F_2(\mu))$ and $U_n(\mu,) = (u(\mu, p_{+1}), u(\mu, p_{-1}))$. By proposition 5 in the same paper [5], adding property 5.5 ensures that $F = I$. Thus, belief updating U_n is Bayesian. $\square$

The mechanism of using F on "shadow" priors and posteriors, and mapping from specified learning rule to a Bayesian one is given in the Figure 2 below taken from Cripps (2018) [5]. F is creating

$$\begin{tikzcd}
\text{Beliefs} \arrow[d, "F"] & \text{Updated Beliefs} \\
\text{Shadow Prior} \arrow[r, "\text{Bayes Updating}"] & \text{Shadow Posterior} \arrow[u, "F^{-1}"']
\end{tikzcd}$$

Figure 2: Bayesian mapping mechanism, Cripps (2018) [5]

Bayesian shadow updating mechanism. If $F = I$, then U_n is itself a Bayesian updating process and posterior beliefs satisfy Bayes' rule. Hence, geodesic learning is Bayesian given priors equal to μ^*. If $p_{+1}^c = p_{-1}^s$, $\mu^* = \frac{1}{2}$. Note that $\mu^* \in [0, 1] \implies p_{+1}^c > \frac{1}{2}, p_{-1}^s < \frac{1}{2} \implies \frac{\partial\mu^*}{\partial p_{-1}^s} < 0$ and $\frac{\partial\mu^*}{\partial p_{+1}^c} > 0$. Because lower μ^* values are conducive to establishing reputation of a commitment type, higher probabilities of truthful reporting of fight outcome decreases μ^* required for Bayesian updating and establishes reputation without equilibrium requirement, as shown in the next section. Similarly, higher value of p_{-1}^s is an impediment to establishing reputation because observing agents will be more likely to learn about incumbent's accommodating behavior. Belief martingale unbiasedness property lies in the core of reputation building in repeated games with a long-run incumbent.

6 Reputation Building

As argued in Watson's seminal paper, in order to establish a reputation of a crazy incumbent, not only do agents need to hold a complete knowledge of the game structure, but also each agent (both short-run and long-run) needs to best-respond to her beliefs. Moreover, the incumbent needs to have 2-nd order knowledge of entrants' best-responding to their beliefs. However, the key conditions are on the belief sets and updating rules. The latter we have already analyzed extensively in the previous section and showed that geodesic learning satisfies Watson's requirements. The major restriction of Watson's argument is that beliefs must not be too dispersed and must be Bayesian [22]. The latter we proved using Cripps propositions 1 and 5 [5], by showing that geodesic learning satisfies all 5 properties.

6.1 Compactness of Beliefs

Because the agents are connected in a random graph, where the number of nodes and edges are unrestricted, we have to implement properties of a random graph to prove properties on belief sets. Tracing properties of random graphs is the key to understanding how signal outcomes propagate through the network of beliefs, affecting immediate neighbors more heavily than the rest. The next sections take on the tasks of analyzing belief bounds on random graphs and exploring the case of the linear network.

There are several results in graph theory that can be applied to geodesic learning, such as average shortest path distance of a random graph being of order $(1 + o(1))\frac{logn}{log\tilde{d}}$, where $\tilde{d}$ is the second-order average degree defined as $\tilde{d} = \frac{\sum w_i^2}{\sum w_i}$, where w_i is the degree of node i in random graph (Chung, Lu (2002) [3]). The next definition is taken from Watson (1993) [22].

Definition 6.1. *Given $\mu \in M$ (belief set) and $r > 0$, let $B_r(\mu) \equiv \{\mu' \in M \quad | \quad d(\mu, \mu') < r\}$ be the ball of radius r centered at μ.*

A set $X \subset M$ is said to be of size k if for each $r > 0$, X can be covered by some $k(r)$ balls of radius r, for some $k : \mathbb{R}_+ \longrightarrow P$, where P is the set of positive integers [22]. An integral part of Watson's argument is to prove that belief conjecture set including the belief sets of all short-run entrants is not too disperse. In other words, the belief set of short-run players is contained in some set of size k. This is also commonly referred to as the repeated game being of size k in Watson (1993) [22]. It is essential for players to hold type $\mathbb{R} - k$ beliefs, as defined by Watson, requiring: 1) both parties involved in the one-period game best-respond to their beliefs, 2) belief sets are of size k, 3) game characteristics is a common knowledge (complete information) [22].

6.2 Geodesic vs. Euclidean Distance

Denote the distribution of shortest path lengths for player i defined on a random graph $G = (n, e)$ as $D_i = (d_{i1}, d_{i2}, ..., d_{in})$. This is the building block of geodesic learning. Denote $T_i = (\tau_{i1}, \tau_{i2}, ..., \tau_{in})$ the euclidean distance distribution of player i from the rest. It is crucial to fix a unit of distance for T: we will normalize it in terms of edge units. Thus, the euclidean distance will be measured in edge units and for simplicity, we normalize all edges having unit length in the underlying graph G.

Theorem 6.2. *T_i first-order stochastically dominates D_i $\forall i$ since $d_{ik} < \tau_{ik} \quad \forall i \quad and \quad k$.*

Proof: Take arbitrary short-run players i, j. Suppose the shortest path connecting them is $(i, x_1, x_2, ..., x_m, j)$ (per definition, a path is a sequence of players leading from i to j via connected nodes). For tractability, we assumed the number of intermediary agents between i and j is m, which implies that $d_{ij} = m + 1$. Construct triangles from consecutive nodes in the following way: $\Delta_1 = (i, x_1, x_2), \Delta_2 = (x_2, x_3, x_4), ..., \Delta_{m-1} = (x_{m-1}, x_m, j)$. The last node of previous triangle is the first vertex of the next one. This guarantees that we cover all vertices and edges on the path from i to j. Denote τ_{ix_j} the euclidean distance from player i to x_j. By edge length normalization, all consecutive players also have unit euclidean distance. Using triangle inequality on $\Delta_1 : \tau_{ix_2} < d_{ix_1} + d_{x_1x_2}$, where d_{ij} is the geodesic distance as defined earlier. Similarly, on triangle $\Delta_2 : \tau_{x_2x_4} < d_{x_2x_3} + d_{x_3x_4}$. Applying this process on all triangles, we are left with the i, j and the last vertex of each triangle at the end of the first elimination round. Note that, even if the initial graph is non-convex, it will become one in finite number of stages after proper deletion of finite vertices through the elimination procedure described above. To summarize, at stage 0, we have $m+2$ vertices and at every next stage the number of vertices left after elimination is at least half of what was before. The process stops when we are left with 3 nodes, i, j and an intermediate node of x_q, such that q is the highest integer satisfying $2^q < m$. Since every triangle in the process is convex, the initial non-convexity does not pose an issue. Thus, at last stages we get:

$$\tau(i, j) < \tau(i, x_q) + \tau(x_q, j) \tag{13}$$

where each of the terms in the right hand side can be expressed as mutually exclusive and exhaustive set of consecutive geodesic distances. $\implies \tau_{ij} < d_{ix_i} + d_{x_1x_2} + ... + d_{x_mj} = d_{ij}$. $\square$

Since $f(\cdot)$ is a decreasing function, $f'(\cdot) < 0 \implies f(\tau(i, j)) > f(d_{ij})$, so we can establish upper bound and k-size-ability on belief sets using euclidean distances.

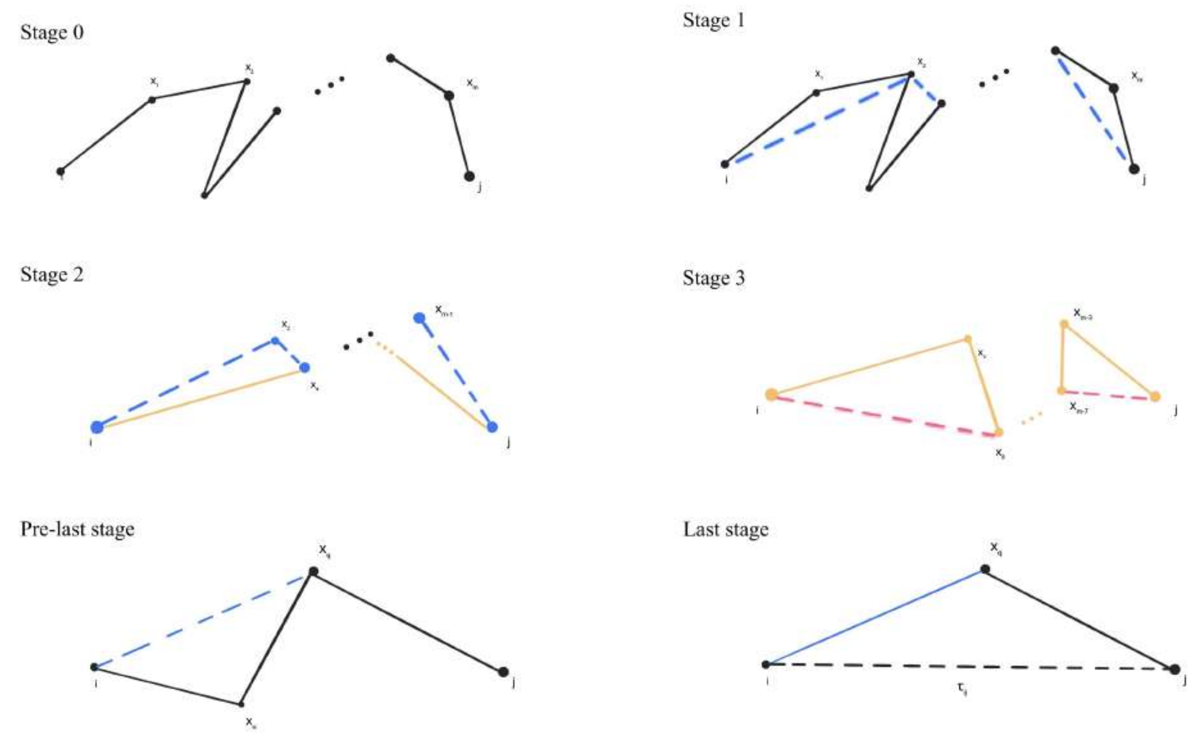

Figure 3: Elimination process for Theorem 6.2

Theorem 6.3. ***Growth Result, Székely (**1997**)** [21]: Let $G = (n, e)$ be a random graph. Let $\Psi_x(n)$ be the set of pairs with euclidean distance equal to x for any x. Then $\#\Psi_x(n) \precsim n^{\frac{4}{3}}$ for any x.*

Proof: This is a known result, but the proof is so beautiful I included it in the Appendices section.

Corollary 6.3.1. *For each agent i, the number of short-run players that are of distance x from i in a random graph, denoted N_x^i, is bounded: $N_x^i \precsim n^{\frac{1}{3}}$*

Proof: Since $\#\Psi_x(n) := \{(i,j) : \tau_{ij} = x\} = \sum_{i \in N} \mathbb{1}(\tau_{ij} = 1) \precsim n^{\frac{4}{3}}$ for any x. Hence, $\sum_{i \in N} N_x^i \precsim n^{\frac{4}{3}}$. Since n is the number of distinct players, $N_x^i \precsim \frac{n^{\frac{4}{3}}}{n} = n^{\frac{1}{3}}$, i. e. $N_x^i \leq c_n n^{\frac{1}{3}}$, for some constant c_n, which depends only on n. Note that the last step is permitted because the operation is of order of magnitude. $\square$

Going back to beliefs: $\mu_i^t = \mu_i^0 + \alpha s_j \sum_{\text{all j played before i}} f(d_{ij}) \leq \mu_{0+\alpha \sum_{\text{all j played before i}} f(\tau_{ij})}$, given the FOSD property of euclidean distance on geodesic one and $s_j \leq 1$. Intuitively, this result restrict the number of immediate neighbors in terms of euclidean distance, as well as the number of neighbors'

neighbors and so on in an arbitrary graph. It is very reasonable to have a large network with well connected agents in which case any signal is spread out promptly and reflected in beliefs almost immediately. In these high-density networks belief can easily reach the upper bound of 1, in which case the entrant's beliefs are converging to the same value, so are not dispersed; however, when networks are sparse or when the parameter value of α or the choice of function $f(\cdot)$ are such that signal transmission is minimized (assuming all entrants start from the same prior required for martingale unbiasedness property), better-connected nodes can accumulate information faster, while sparse components of the graph can be left with minimal signals, thus leading to significant heterogeneity in posterior beliefs. Theorem 6.3 is what will help to show that even in these cases beliefs are not much different from each other and can be covered by k balls for some k and ball centers $\mu_1, \mu_2, ..., \mu_k$, since it restricts the frequency of node's immediate signals, its neighborhood's immediate signals and so on.

$$\mu_i^t = \mu^0 + \alpha \sum_{\text{all} j \text{played before} i} f(\tau_{ij}) \precsim \mu^0 + \alpha n^{\frac{1}{3}} [f(\lambda_1) + f(\lambda_2) + ... + f(\lambda_\gamma)] \tag{14}$$

where γ is the number of distinct euclidean distances and $\lambda_1, \lambda_2, ..., \lambda_\gamma$ are those distances between i and other short-run entrant (each repeating $\precsim n^{\frac{1}{3}}$ times). The number of distinct distances is also bound from above, and is the famous Erdős–Rényi distinct distances problem solved in 2010. With appropriate choice of α and $f(\cdot)$ beliefs for i and j can be made arbitrarily close to each other such that $\mu_i^t - \mu_j^t < r(n)$ for some $r(\cdot) > 0$ $\forall n$ and period $t \implies$ the belief set of entrants' conjectures can be covered by k balls where k is a function of threshold r, which is a function of $n \implies k : \mathbb{R}_+ \longrightarrow P$. Since the belief set and also the game are of size k, we can now use Watson's theorem in our framework.

Theorem 6.4. ***Watson*** 1993 *[22]: Take any system of conjectures $\mu \in M$ of short-run entrants in Bayesian entry-deterrence repeated game with $\mu^0 \geq \epsilon$, and take any outcome history ω^t. Then under conditions of 1) Bayesian updating rule, 2) Best-responding agent, 3) size-k complete information game,*

$$\#\{\mu^h < z | h \in \textit{arbitrary outcome distribution}\} < \frac{ln\epsilon}{lnz} \tag{15}$$

for every $z \in (0, 1)$.

The incumbent manages to successfully establish a reputation because the number of periods in which entrants believe that the probability of the incumbent behaving as the sane type is characterized by an upper bound, which depends on k and ϵ, the former depending on n, the number of short-run entrants. Reputation is established even when agents observe outcomes in a spatially discounted manner and update their beliefs heterogeneously. Note that our framework does not destroy the reputation of the long-run incumbent after a single period of accommodation outcome, unlike other models, treating both game outcomes of fight and accommodate as signals of equal magnitude but opposite signs. In dense networks, this reputation is obviously established faster. Because geodesic learning is analytically hard to tackle in Perfect Bayesian Equilibrium, by establishing core properties on the framework, we deduce that for establishing a reputation the system does not need to be in equilibrium. It is enough for agents to best-respond to their beliefs, making the model both more reflective of market entry game (via heterogeneous beliefs and observing) and analytically tractable.

7 The Linear Network

Simulations unveil and interesting phenomenon of intermittent entry. This is more expressed and appealing to explore in the linear case. Let $G = (n, e)$ be a linear graph on set of players $N = \{1, 2, ..., n\}$ and $n-1$ edges, where only consecutive agents are connected with an edge. All entrants will observe the outcome of all period games before they are called to play, because G is a connected graph, i. e. there exists a path form any node to any other one via finite number of edges. Throughout this analysis, we will assume that entrants play successively, player 1, 2, and so on. However, since our proposed framework introduced heterogeneity in signal absorption over the graph, immediate neighbors of the player in the current period will receive the "strongest" signal (the least spatially discounted signal of the outcome) and will be more likely to stay out when they are called to play (if the signal was $+1$). What we observe is that most of the entrants will stay out, because by the time they are called to play they would have accumulated enough signals to update their beliefs above some threshold and stay out by best-responding to that (assuming normal type chases to establish reputation by choosing Stackelberg action of fighting each period, especially in the beginning of the game). However, staying out means that the player transmits no information for later entrants and here rises a tradeoff of whether the agents (especially in initial stages) who have updated their beliefs enough so that their best-response is to stay out, should nevertheless enter and transmit signal for the benefit of upcoming players rather than staying out and being better off. Perhaps one can model this as a cooperative game of entrants' collusion, who care about their total payoff from the perspective of a social planner. In this case, the agent will avoid staying out in initial stages and will help to accumulate enough signals for the later entrants with positive probability. Moreover, the entrants can possibly disincentivize the incumbent to fight in initial stages and because of discounting, can also disincentivize establishing reputation of a crazy type at later stages. However, a myopic entrant will always best-respond to the one-period game and maximize her one-period payoff, opting to stay out rather than sacrificing her payoff to the good of total signal accumulation for entrants in upcoming stages. With short-lived myopic entrants, market entry gaps will be observed, which become lengthier over time. Meanwhile, the intermittent entries are the agents, who have not managed to pass the belief threshold (due to lack of signals or connectivity) and think that the incumbent will fight with low probability ($\mu_i^t < p = b$,as shown previously in section 4). Intuitively, the signals have been discounted so heavily when reaching these agents (nodes) that they did not consider them "seriously" (or credible threats) and did not update their beliefs significantly. The length of these gaps will depend on the number of nodes and edges, α, and $f(\cdot)$ as shown below. The very last players in the sequence of entrants receive the most signals: however, since most agents do not enter market by Theorem 6.3, they (and all other nodes) receive the signals at much much fewer periods that one could conjecture initially. Denote $i_1, i_2, ..., i_\beta$ the entrants who entered at the end of the repeated game. Denote the threshold of entry belief as p (this can be determined from payoff matrices in the first section). Because these players entered, at the time they were called to play their belief of commitment type was lower than p.

Definition 7.1. *The number of entrants who stayed out between consecutive entrants i_l and i_{l+1} is called gap size and denoted s_l.*

Assume i_1, i. e. the first entrant, is always the player in period 1 ($\mu^0 < p$, otherwise the game will proceed with no entry and signals). For characterizing gap lengths we can assume that the incumbent's best-response is to fight in all stages. Simulations in section 9 are also consistent with the fact that payoff is also optimized by choosing Stackelberg payoff of the commitment type in all stages. Thus, we can assume a signal of $s_j^t = +1$ for any playing entrant j and period t. Since s_1, and s_2 are the gap sizes after first and second entries, we have:

$$\begin{gathered}\mu_{i_2}^{s_1+2} = \mu^0 + \alpha f(s_1+1) < p \\ \mu_{i_2-1}^{s_1+1} = \mu^0 + \alpha f(s_1) > p \\ \mu_{i_2+1}^{s_1+3} = \mu^0 + \alpha[f(1) + f(s_1+2)] > p\end{gathered} \tag{16}$$

Note that player $i_2 + 1$ accumulates two signals, from i_1 and i_2. Note that it is enough to look at the previous and next players around each entry to characterize the gap length. Assume, $f(x) = \frac{1}{x}, f'(x) < 0$.

$$\begin{aligned} f(s_1+1) < \frac{p-\mu^0}{\alpha} &\implies \frac{1}{s_1+1} < \frac{p-\mu^0}{\alpha} \implies s_1 > \frac{\alpha - p + \mu^0}{p-\mu^0} \\ f(s_1) > \frac{p-\mu^0}{\alpha} &\implies \frac{1}{s_1} < \frac{p-\mu^0}{\alpha} \implies s_1 < \frac{\alpha}{p-\mu^0} \\ f(s_1+2) + f(1) > \frac{p-\mu^0}{\alpha} &\implies \frac{1}{s_1+2} > \frac{p-\mu^0-\alpha}{\alpha} \implies s_1 < \frac{\alpha - 2p + 2\mu^0 + 2\alpha}{p - \mu^0 - \alpha}\end{aligned} \tag{17}$$

giving, $\frac{\alpha-p+\mu^0}{p-\mu^0} < s_1 < \frac{\alpha}{p-\mu^0} = s_1^*$, requiring $\alpha - p + \mu^0 < \alpha \implies p > \mu^0$, i. e. threshold needs to be higher than the initial beliefs, which we specified earlier. As simulations show, the gaps between market entries becomes wider over time. Two forces are accountable for this effect: 1) over time later agents receive more signals, increasing belief value (assuming signal is +1 and contributing to less entry, 2) later agents have inferior geodesic distribution in linear network (recall that we assumed that agents enter successively) meaning the signals they received are discounted heavily, leading to the opposite effect of lagging belief updates and inclination towards entry. As our results show, the first effect is more dominant. $\frac{\partial s_1^*}{\partial \alpha} = \frac{1}{p-\mu^0} > 0$, since $p > \mu^0$, and $\frac{\partial s_1^*}{\partial p} = \frac{-1}{(p-\mu^0)^2} < 0$, i. e. the higher the threshold, the smaller the gap size upper bound and hence the gap size, since less players will have enough signals to update their beliefs above the threshold, inducing more frequent entry. Similarly,

$\frac{\partial s_1^*}{\partial \mu^0} = \frac{1}{(p-\mu^0)^2} > 0$. Higher prior of commitment type already induces less entry and larger gap sizes. For gap size after the second entrant, we have

$$\begin{gathered}
\mu_{i_3}^{s_1+s_2+2} = \mu^0 + \alpha[f(s_1+s_2+2) + f(s_2+1)] < p \implies \frac{2s_2+s_1+3}{(s_2+1)(s_1+s_2+2)} < \frac{p-\mu^0}{\alpha} \\
\mu_{i_3-1}^{s_1+s_2+1} = \mu^0 + \alpha[f(s_1+s_2+1) + f(s_2)] > p \implies \frac{2s_2+s_1+1}{s_2(s_1+s_2+1)} > \frac{p-\mu^0}{\alpha} \\
\mu_{i_3+1}^{s_1+s_2+3} = \mu^0 + \alpha[f(s_1+s_2+3) + f(s_2+2) + f(1)] > p \implies \\
\frac{2s_2+s_1+5+s_1s_2+2s_1+s_2^2+5s_2+6}{(s_2+2)(s_1+s_2+3)} > \frac{p-\mu^0}{\alpha}
\end{gathered} \tag{18}$$

We need a lower bound on s_2, so we will use the first equation of the system, giving

$$\begin{gathered}
2s_2\alpha + s_1\alpha + 3\alpha < ps_1s_2 + ps_1 + ps_2^2 + ps_2 + 2ps_2 + 2p - \mu^0 s_1 s_2 - \mu^0 s_1 - \mu^0 s_2^2 - \mu^0 s_2 - 2\mu^0 s_2 - 2\mu^0 \implies \\
s_2^2(p-\mu^0) + s_2(3p - 3\mu^0 + s_1 p - s_1\mu^0 - 2\alpha) + 2p - 2\mu^0 + ps_1 - \mu^0 s_1 - s_1\alpha - 3\alpha > 0
\end{gathered} \tag{19}$$

with only acceptable solutions with $s_1, s_2 > 0$ being $s_2 > \frac{1}{2}[\sqrt{\frac{4\alpha^2+(s_1+1)^2(p-\mu^0)^2}{(p-\mu^0)^2}} - \frac{2\alpha}{\mu^0-p} - s_1 - 3] = \xi$. Now, $s_1 < \xi \iff s_1 \le \frac{3\alpha}{2(p-\mu^0)} - 2$. Recall, $s_1 > \frac{\alpha-p+\mu^0}{p-\mu^0}$, hence for $s_1 < s_2$ we require $\frac{3\alpha}{2(p-\mu^0)} - 2 > \frac{\alpha-p+\mu^0}{p-\mu^0} \implies \alpha > 2(p-\mu^0)$. Alternatively, we had that $s_1 < \frac{\alpha}{p-\mu^0}$. If $\xi \ge \frac{\alpha}{p-\mu^0}$, then $s_2 > s_1$. Hence,

$$\begin{gathered}
\xi \ge \frac{\alpha}{p-\mu^0} \implies s_1 \le \frac{\alpha^2 - 2(\mu^0)^2 + 4\mu^0 p - 2p^2}{(\mu^0-p)^2} \iff \\
\frac{\alpha}{p-\mu^0} \le \frac{\alpha^2 - 2(\mu^0)^2 + 4\mu^0 p - 2p^2}{(\mu^0-p)^2} \implies \\
\mu^0 < p \le \frac{1}{2}(\alpha + 2\mu^0)
\end{gathered} \tag{20}$$

which is equivalent to above expression of $\alpha > 2(p - \mu^0)$. Intuitively, this means that gaps will become wider, i. e. less and less short-run players will enter the market if there is significantly enough signal diffusion between agents, reflected in parameter α. This is a lower bound on α, which nonetheless does not pose a concern to beliefs being of size-k in the argument of the previous section, because, firstly, this lower bound is for a specific choice of function f, and secondly, the choice of $f(\cdot)$ was not fixed in previous proofs. Thus, this bound does not induce beliefs to be too dispersed.

8 On General Graphs

It is clear that the properties of graphs translate to properties of the model, such as gap length, learning speed or belief compactness. The distribution of edges on n nodes defines the geodesic probability distribution. Adding new edges on G, creates a new graph G' whose geodesic distance distribution first-order stochastically dominates that of G's, while rearranging those introduces second-order stochastic domination (Goyal (2007) [12]).

Proposition 8.1. *Let $G = (n, e)$ be network and $G' = G + g_{ij}$ (adding new edge between nodes i and j which did not exist on G). Denote D as the geodesic distance cumulative distribution of G and D' that of G'. Then the following are equivalent: 1) D' FOSD D, 2) $\mu_i^t|_{G'} > \mu_i^t|_G$ $\forall$ i and t, assuming consistent signal of $s = +1$, 3) Reputation is established faster on G'.*

Proof: Note that geodesic distances for any pair of arbitrary nodes cannot be higher in G' than in G. Adding g_{ij} makes traveling from some node to another through the shortest path is shorter than before. When g_{ij} is added, d_{ik} for some random node k is: $d_{ik}|_{G'} = min(d_{ik}|_G, d_{jk}|_{G'} + 1)$. Hence, $d_{ik}|_{G'} \geq d_{ik}|_G$ for any pair (i, k). Because the choice of (i, k) is arbitrary $\implies$ D' FOSD D. Consequently, $f(d_{ik}|_{G'}) \geq f(d_{ik}|_G) \implies \mu_i^t|_{G'} \geq \mu_i^t|_G$. Because this argument is equivalent to having higher α, and $\frac{\partial s_1^*}{\partial \alpha} > 0$, then the gap size is higher on G' too. This is intuitive since adding more edges establishes more channels for signal transmission and enables more agents to update their beliefs, who were previously receiving no or weak signals. These agents will manage to update their beliefs above the threshold and will not enter, increasing the gap size and decreasing the number of entries throughout the game. $\square$

Table 3: α and threshold belief fixed, $f(\cdot)$ changes. Results shown for different n.

Nodes	α	Threshold belief ($= b$)	Lowest optimal number of fight stages	Maximized payoff	% from Stackelberg payoff	Lowest optimal number of fight stages	Maximized payoff	% from Stackelberg payoff
			$f(x) = 1/x$			$f(x) = e^{-x}$		
5	0.3	0.6	4	4	40	3	2	20.0
10	0.3	0.6	9	11	55.0	9	5	25.0
15	0.3	0.6	9	19	63.3	13	7	23.3
20	0.3	0.6	15	28	70.0	19	10	25.0
25	0.3	0.6	22	35	70.0	23	12	24.0
35	0.3	0.6	30	52	74.3	33	17	24.3
50	0.3	0.6	47	76	76.0	49	25	25.0
65	0.3	0.6	56	101	77.7	63	32	24.6
75	0.3	0.6	65	118	78.7	73	37	24.7
90	0.3	0.6	85	144	80.0	89	45	25.0
100	0.3	0.6	95	161	80.5	99	50	25.0
125	0.3	0.6	115	205	82.0	123	62	24.8
150	0.3	0.6	148	246	82.0	149	75	25.0
250	0.3	0.6	241	422	84.4			
300	0.3	0.6	290	510	85.0			

9 Simulations and Results

9.1 Best-responding Simulations on Linear Graphs

Simulations with best-responding agents were run on several specifications of $\alpha, f(\cdot)$, as well as graph properties, such as the number of nodes and edges. The long-run player engages with short-run agents consecutively starting from agent 1. Since for the general networks simulation section the graph is created randomly, the choice of short-run entrants' sequence is irrelevant. Perfectly patient agents ($\delta = 1$) were assumed in all simulations, as well as an initial belief of $\mu^0 = 0.5$.

Tables $3 - 5$ in this section are for linear graphs. Table 3 shows the results when the transmission function $f(\cdot)$ is changed. $f(x) = e^{-x}$ is decreasing faster on the set of positive integers, hence for all numbers of nodes (players) the incumbent establishes reputation at later periods (lowest optimal number of fight stages is higher for $f(x) = e^{-x}$ except for $n = 5$). The same trend is noticed for the maximized cumulative payoff of the incumbent. As discussed, the Stackelberg payoff from each period is $a = 2$ and this is the best that a commitment type (or a sane type mimicking the commitment type) can do. As the number of players goes up (or the number of time periods goes up), the incumbent achieves a higher proportion of its cumulative Stackelberg payoff by fighting in optimal number of stages.

Table 4: Threshold belief and $f(\cdot)$ fixed, α changes. Results shown for different n.

Nodes	Threshold belief (= b)	$f(\cdot)$	Lowest optimal number of fight stages	Maximized payoff	% from Stackelberg payoff	Lowest optimal number of fight stages	Maximized payoff	% from Stackelberg payoff
				$\alpha = 0.3$			$\alpha = 0.1$	
5	0.6	1/x	4	4	40.0	4	1	10.0
10	0.6	1/x	9	11	55.0	9	5	25.0
15	0.6	1/x	9	19	63.3	12	10	33.3
20	0.6	1/x	15	28	70.0	18	16	40.0
25	0.6	1/x	22	35	70.0	14	20	40.0
35	0.6	1/x	30	52	74.3	34	31	44.3
50	0.6	1/x	47	76	76.0	46	50	50.0
65	0.6	1/x	56	101	77.7	62	70	53.8
75	0.6	1/x	65	118	78.7	74	81	54.0
90	0.6	1/x	85	144	80.0	96	100	55.6
100	0.6	1/x	95	161	80.5	98	113	56.5
125	0.6	1/x	115	205	82.0	124	145	58.0
150	0.6	1/x	148	246	82.0	147	180	60.0
250	0.6	1/x	241	422	84.4			
300	0.6	1/x	290	510	85.0			

Table 4 shows linear simulation results when the value of α is changed. These results are consistent with previous analytical outcomes, showing that a higher value of α leads to faster reputation effects; moreover, the difference in optimal fight stages necessary to achieve the highest payoff is more noticeable for higher values of n (more on this elaborated in the next section). The maximized payoff and % from Stackelberg payoff are consistently lower for $\alpha = 0.1$.

In Table 5, the threshold belief is changed. This is the belief level, above which the entrant will best-respond by not entering the market. As shown earlier and given that entrants are myopic (so are maximizing payoffs for a single period only), this threshold value is equivalent to b in the payoff matrix. A higher b makes market entry easier and reputation effects harder to establish. The results in Table 5 are consistent with my conjectures. In fact, for $n \leq 65$, the incumbent does not even manage to establish a reputation and is left with 0 payoff when $b = 0.8$. Maximized payoffs % from Stackelberg payoff are consistently lower for $b = 0.8$.

Cumulative payoff patterns are shown in Figures 4 and 5 below, confirming analytical comparative statics results derived in earlier sections, with increasing α and decreasing threshold belief values associated with faster reputation establishments and higher maximized payoffs. Note that two figures are shown for different values of $n = 75$ and $n = 50$. The patterns are very similar and the only effect of increased number of players is the level effect in payoff. Higher n linear networks enable more

Table 5: α and $f(\cdot)$ fixed, threshold belief changes. Results shown for different n.

Nodes	α	$f(\cdot)$	Lowest optimal number of fight stages	Maximized payoff	% from Stackelberg payoff	Lowest optimal number of fight stages	Maximized payoff	% from Stackelberg payoff
			$b = 0.8$ (threshold belief)			$b = 0.6$ (threshold belief)		
5	0.1	1/x	0	0	0.0	4	1	10.0
10	0.1	1/x	0	0	0.0	9	5	25.0
15	0.1	1/x	0	0	0.0	12	10	33.3
20	0.1	1/x	0	0	0.0	18	16	40.0
25	0.1	1/x	0	0	0.0	14	20	40.0
35	0.1	1/x	0	0	0.0	34	31	44.3
50	0.1	1/x	0	0	0.0	46	50	50.0
65	0.1	1/x	0	0	0.0	62	70	53.8
75	0.1	1/x	73	4	2.7	74	81	54.0
90	0.1	1/x	89	12	6.7	96	100	55.6
100	0.1	1/x	99	17	8.5	98	113	56.5
125	0.1	1/x	123	29	11.6	124	145	58.0
150	0.1	1/x	149	42	14.0	147	180	60.0

maximized payoff for the incumbent.

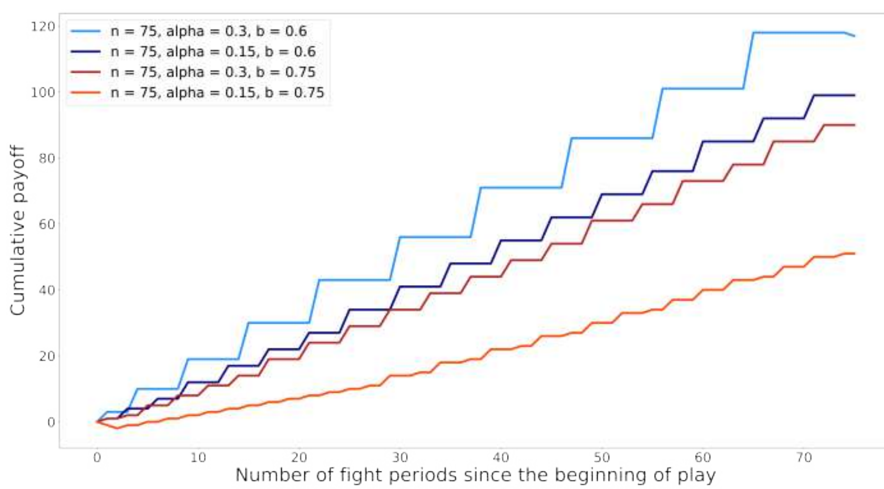

Figure 4: Payoff pattern for linear network, fixed $n = 75$, varying $alpha$ and threshold belief ($p = b$).

9.2 Market Entry Simulations on Linear Networks

Market entry simulations were run on linear networks. Table 6 varies number of players. The first two rows, however, show varying levels of the initial belief, confirming the analytical results stating that higher μ^0 results in less entries (7.4 % down from 27.7 %). Linear networks induced on lower number of players result in harder reputation establishment, have higher market entry %, as shown in the last column.

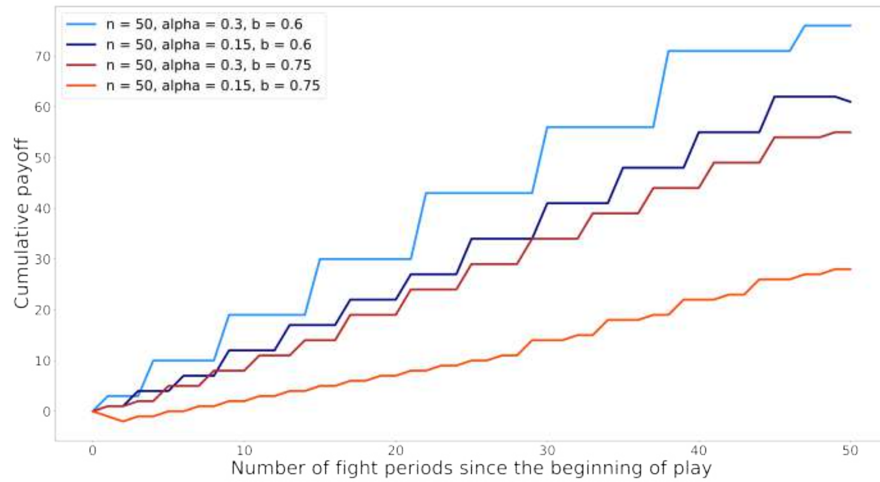

Figure 5: Payoff pattern for linear network, fixed $n = 50$, varying $alpha$ and threshold belief ($p = b$).

Table 6: α, $f(\cdot)$ fixed and threshold belief fixed. Results shown for different n.

Nodes	α	Threshold belief ($= b$)	Initial belief ($= \mu^0$)	$f(\cdot)$	Number of entries	Entry %
1500	0.3	0.6	0.1	1/x	415	27.7
1500	0.3	0.6	0.5	1/x	111	7.4
900	0.3	0.6	0.5	1/x	73	8.1
800	0.3	0.6	0.5	1/x	66	8.3
700	0.3	0.6	0.5	1/x	59	8.4
600	0.3	0.6	0.5	1/x	52	8.7
500	0.3	0.6	0.5	1/x	45	9.0
200	0.3	0.6	0.5	1/x	22	11.0

Table 7 fixes $n = 500$ and $p = b = 0.6$ to test results in section 7. Entry % goes down as α goes up. Moreover, as shown in section 7, higher α values also induce increasing sizes of gaps. The higher the value of α, the faster signals spread, the less entrants actually enter the market and hence the higher the gap size. Figure 6 shows the gap sizes for varying levels of α after k-th entry (shown on x axis). Eventually, the gap size stops growing for each α, while the level effects between different levels of α are obvious. Figure 7 also shows a chronological market entry visualization. As conjectured, with increasing levels of α parameter: 1) the gap size goes up, 2) market entries become more sparse, 3) within each α value, the gaps tend to grow as well. Moreover, the higher the parameter value, the more the tendency for the gap size to grow in the beginning of the repeated game, as shown in equation (20) for gap sizes 1 and 2. To sustain increasing gap sizes for later periods, more restrictive conditions than (20) are needed on α.

Table 7: n, $f(\cdot)$ fixed and threshold belief fixed, α changes.

Nodes	α	Threshold belief ($= b$)	Initial belief ($= \mu^0$)	$f(\cdot)$	Number of entries	Entry %
500	0.1	0.6	0.5	1/x	108	21.6
500	0.2	0.6	0.5	1/x	62	12.4
500	0.3	0.6	0.5	1/x	45	9
500	0.4	0.6	0.5	1/x	36	7.2
500	0.5	0.6	0.5	1/x	31	6.2
500	0.6	0.6	0.5	1/x	27	5.4
500	0.7	0.6	0.5	1/x	24	4.8
500	0.8	0.6	0.5	1/x	21	4.2
500	0.9	0.6	0.5	1/x	20	4
500	1	0.6	0.5	1/x	18	3.6

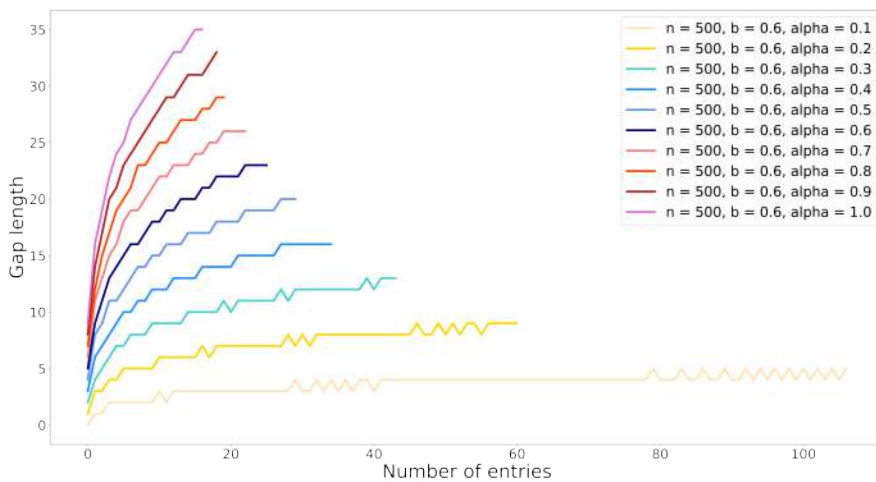

Figure 6: Gap size for linear network, fixed n, varying $alpha$. Horizontal axis shows the k-th entry.

9.3 Iterated Simulations on General Networks

In each case, a random graph was generated by specifying the number of nodes n and generating edges between randomly chosen nodes e times. The entrant list is consecutive, fixed as before. Because the graphs are generated randomly, simulations were repeated $5, 10, 15$ times. Results are shown in Tables $8 - 10$.

Table 8 corroborates the main outcomes established earlier. The most interesting outcome that is revealed here (also present in previous tables but less evident) is the following: when threshold belief goes up to 0.8 in sparse networks, the optimal number of periods to fight is 0, i. e. the incumbent is immediately discouraged to mimic the crazy type. As the graph becomes denser, it starts to become reasonable to chase to establish a reputation for the incumbent, even though it might take long periods

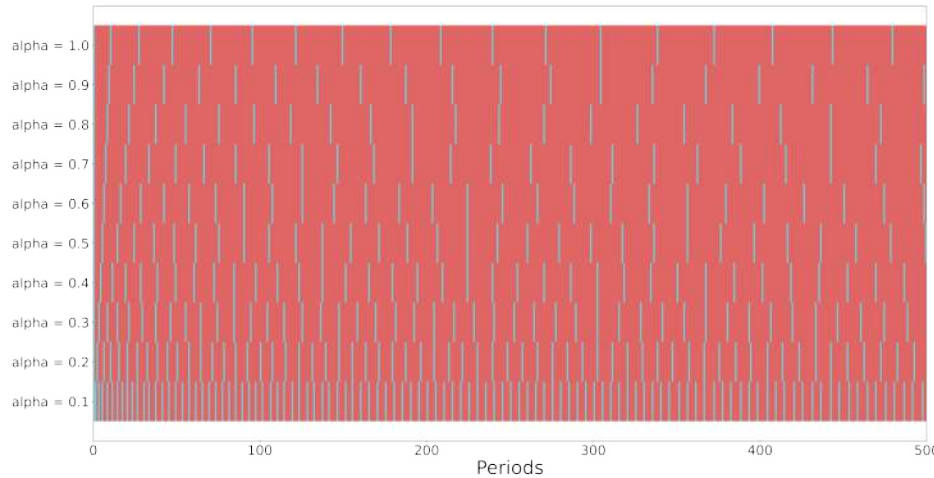

Figure 7: Chronological market entry simulation for fixed n and varying α

to do so. Meanwhile, for low threshold values, it is reasonable to start fighting even in sparse networks. This is why we see that the optimal number of fight stages is higher for $b = 0.6$ compared to $b = 0.8$ in the first rows of Table 8. As the density of graph goes up, the optimal number of fight periods goes down significantly for $b = 0.6$ and becomes lower than that of $b = 0.8$. Alternatively, if lowest optimal number of fight stages is less than 1, one can read it as if it is higher than n (because the available n time periods are simply not enough to establish a reputation by fighting). With this mental replacement, lowest optimal number of fight stages column for $b = 0.8$ is always higher than that of $b = 0.6$. This result is also shown in Figure 8.

Table 9 provides closer look at comparison when the number of nodes is changing. It is critical to observe that initially lower n networks perform better, but lag towards the end of the table as number of edges increases. I conjecture that there is a saturation point for each n, to which the maximized payoff and % from Stackelberg payoff converges. The higher the n, the higher this upper bound. This can explain why % from Stackelberg payoffs swap midway for $n = 50$ and $n = 75$.

Table 10 concludes that our results are robust to different values of iterations. The results are more convergent for higher number of edges, i. e. for more dense networks. For all iteration values, the incumbent manages to get up to 94.6 % of the possible Stackelberg payoff.

Iterations	Edges	α	$f(\cdot)$	Lowest optimal number of fight stages	Maximized payoff	% from Stackelberg payoff	Lowest optimal number of fight stages	Maximized payoff	% from Stackelberg payoff	Lowest optimal number of fight stages	Maximized payoff	% from Stackelberg payoff	Lowest optimal number of fight stages	Maximized payoff	% from Stackelberg payoff
				$n = 75$						$n = 50$					
				$p = 0.6$			$p = 0.8$			$p = 0.6$			$p = 0.8$		
10	15	0.3	1/x	4.0	2.0	1.3	0.0	0.0	0.0	14.4	6.4	6.4	0.0	0.0	0.0
10	30	0.3	1/x	29.8	23.5	15.7	0.5	0.1	0.1	17.4	37.9	37.9	23.7	11.1	11.1
10	45	0.3	1/x	27.9	54.2	36.1	37.8	24.4	16.3	13.3	64.5	64.5	21.4	45.4	45.4
10	60	0.3	1/x	32.3	86.3	57.5	32.8	64.2	42.8	6.8	78.6	78.6	18.8	66.0	66.0
10	75	0.3	1/x	30.9	110.5	73.7	31.8	88.5	59.0	6.3	85.8	85.8	18.5	77.1	77.1
10	100	0.3	1/x	10.2	128.5	85.7	31.4	112.2	74.8	2.7	91.9	91.9	18.1	83.2	83.2
10	125	0.3	1/x	6.5	136.8	91.2	28.3	124.5	83.0	3.2	93.6	93.6	14.7	86.6	86.6
10	150	0.3	1/x	4.7	141.1	94.1	17.3	130.1	86.7	2.9	93.1	93.1	10.4	87.8	87.8
10	175	0.3	1/x	2.4	142.8	95.2	26.2	134.2	89.5	3.6	94.0	94.0	12.8	88.3	88.3
10	200	0.3	1/x	2.6	143.4	95.6	13.3	135.7	90.5	3.8	94.0	94.0	17.0	89.2	89.2
10	225	0.3	1/x	2.2	143.4	95.6	5.6	137.8	91.9	7.0	94.0	94.0	21.8	89.2	89.2
10	250	0.3	1/x	3.2	143.8	95.9	16.4	137.2	91.5	12.6	94.0	94.0	4.2	90.4	90.4
10	275	0.3	1/x	2.6	143.8	95.9	9.2	138.0	92.0	14.1	94.0	94.0	4.9	90.2	90.2
10	300	0.3	1/x	3.0	144.0	96.0	16.0	138.0	92.0	14.0	94.0	94.0	3.4	91.0	91.0
10	325	0.3	1/x	2.3	144.0	96.0	15.0	138.0	92.0	10.5	94.3	94.3	4.2	91.0	91.0

Table 8: Simulation results for general randomly generated graphs. The number of nodes in all results is 500. The number of edges and hence the density of graph is varying, as well as the threshold belief ($= b$). α is fixed. 10 iterations were completed and the results were averaged out.

Nodes	Edges	α	Threshold belief (=b)	$f(\cdot)$	Lowest optimal number of fight stages	Maximized payoff	% from Stackelberg payoff	Lowest optimal number of fight stages	Maximized payoff	% from Stackelberg payoff
					$n = 75$			$n = 50$		
10	15	0.1	0.8	1/x	0.0	0.0	0.0	0.0	0.0	0.0
10	30	0.1	0.8	1/x	0.0	0.0	0.0	2.5	0.1	0.1
10	45	0.1	0.8	1/x	8.0	0.9	0.6	29.3	16.1	16.1
10	60	0.1	0.8	1/x	47.9	22.1	14.7	28.7	43.8	43.8
10	75	0.1	0.8	1/x	44.6	54.5	36.3	30.5	53.6	53.6
10	100	0.1	0.8	1/x	44.4	80.9	53.9	31.5	62.2	62.2
10	125	0.1	0.8	1/x	46.0	97.6	65.1	26.9	68.0	68.0
10	150	0.1	0.8	1/x	41.6	106.2	70.8	25.3	71.4	71.4
10	175	0.1	0.8	1/x	38.6	111.7	74.5	22.7	72.4	72.4
10	200	0.1	0.8	1/x	36.4	115.3	76.9	22.4	74.3	74.3
10	225	0.1	0.8	1/x	39.6	117.0	78.0	26.6	75.1	75.1
10	250	0.1	0.8	1/x	40.3	119.7	79.8	18.7	77.5	77.5
10	275	0.1	0.8	1/x	38.5	120.4	80.3	16.9	77.7	77.7
10	300	0.1	0.8	1/x	26.8	121.8	81.2	22.5	78.3	78.3
10	325	0.1	0.8	1/x	35.2	123.0	82.0	14.9	79.0	79.0

Table 9: Simulation results on randomly generated graphs for varying edge numbers (density). $n = 75$ and $n = 50$ simulations were run. All other parameters are fixed.

Nodes	Edges	α	Threshold belief (=b)	$f(\cdot)$	Lowest optimal number of fight stages	Maximized payoff	% from Stackelberg payoff	Lowest optimal number of fight stages	Maximized payoff	% from Stackelberg payoff	Lowest optimal number of fight stages	Maximized payoff	% from Stackelberg payoff
					$iterations = 5$			$iterations = 10$			$iterations = 15$		
50	15	0.3	0.6	1/x	8.4	4.8	4.8	6.8	7.5	7.5	9.5	8.4	8.4
50	30	0.3	0.6	1/x	22.2	38.0	38.0	18.0	37.2	37.2	20.1	37.3	37.3
50	45	0.3	0.6	1/x	14.4	65.0	65.0	13.9	67.1	67.1	15.1	65.7	65.7
50	60	0.3	0.6	1/x	7.4	81.8	81.8	18.7	81.0	81.0	13.7	78.9	78.9
50	75	0.3	0.6	1/x	9.6	86.2	86.2	8.0	87.3	87.3	11.2	87.7	87.7
50	100	0.3	0.6	1/x	3.2	92.0	92.0	2.5	92.2	92.2	2.3	91.5	91.5
50	125	0.3	0.6	1/x	3.0	91.8	91.8	2.4	92.4	92.4	3.0	93.1	93.1
50	150	0.3	0.6	1/x	3.4	93.6	93.6	2.6	94.0	94.0	2.5	93.7	93.7
50	175	0.3	0.6	1/x	4.4	94.0	94.0	4.9	94.0	94.0	4.9	94.0	94.0
50	200	0.3	0.6	1/x	5.0	94.0	94.0	4.3	94.0	94.0	5.3	93.9	93.9
50	225	0.3	0.6	1/x	12.8	94.0	94.0	7.0	94.0	94.0	5.1	94.0	94.0
50	250	0.3	0.6	1/x	6.0	94.0	94.0	5.7	94.0	94.0	8.5	94.0	94.0
50	275	0.3	0.6	1/x	5.6	94.0	94.0	13.4	94.1	94.1	8.1	94.0	94.0
50	300	0.3	0.6	1/x	15.8	94.0	94.0	15.5	94.6	94.6	10.0	94.6	94.6
50	325	0.3	0.6	1/x	10.8	94.6	94.6	14.4	94.6	94.6	12.5	94.6	94.6

Table 10: Simulation results with different number of iterations used.

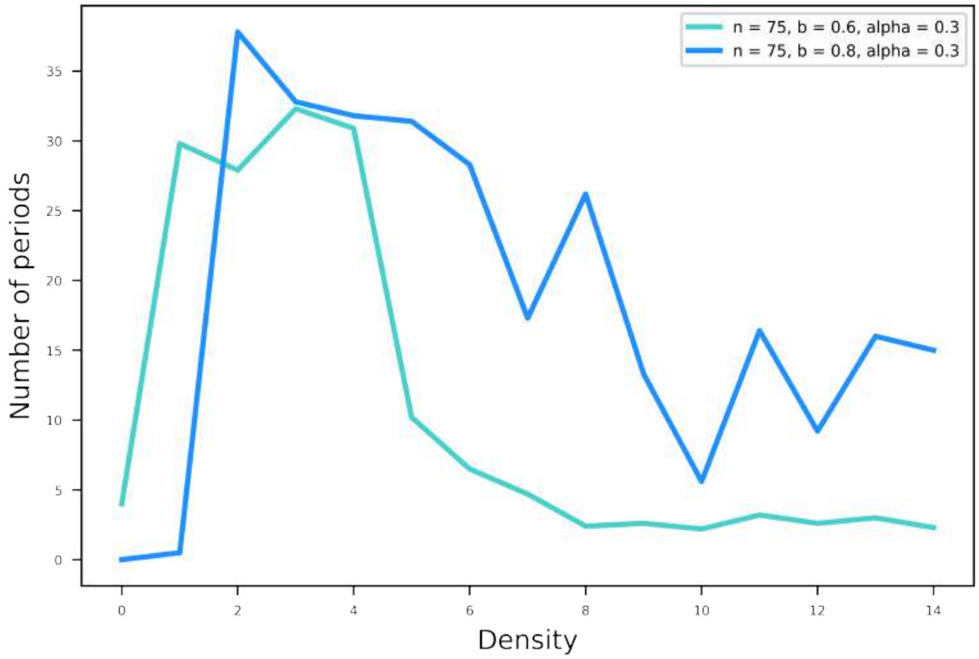

Figure 8: Lowest optimal number of fight periods, fixed n, α and varying threshold belief.

10 Conclusion

I have introduced an interesting geodesic belief updating (learning) rule, which can be mapped into a Bayesian one. Using results from graph theory, I showed that belief conjectures of all entrants are not too disperse and can be covered by k-balls. Hence, Watson's seminal result [22] with only best-responding agents can be applied to my framework with no equilibrium concept needed. I have characterized market entry and gap sizes, ran simulations with varying parameters. Simulation results were consistent with analytical conjectures, showing: 1) in linear graphs, increasing the number of players ceteris paribus increases % of maximum possible total Stackelberg payoff achievable by the long-run incumbent, 2) in linear graphs, increasing α when b is fixed, on average makes establishing reputation faster with lower optimal number of fight stages needed (Table 4) and higher payoffs, 3) in linear graphs, increasing b when α is fixed, makes reputation effects harder to establish and less profitable (Table 5), 4) in linear graphs, market entry is suppressed when any of n, μ^0 or α go up ceteris paribus, 5) in general graphs, establishing reputation is faster/easier when b goes down, graph becomes denser (number of edges goes up) or α goes up all else equal. The value of n is volatile in determining trends in payoff levels.

11 Discussion

The proposed model is a reasonable extension of the Kreps model [16]. The latter assumes that all entrants observe the previous play histories in the same way, which is not a realistic assumption. This is the reason why there also has been a lot of research in opinion formation and influence in social networks, such as the DeGroot modeling, which established certain conditions on the opinion weight matrix in order for the network to reach consensus, or have aggregation or diffusion properties. This model shifts the perspective from the time horizon to the spatial dimension induced by underlying graph properties. Thus, this model builds up on the Kreps and Wilson model by adding another dimension of learning - i. e. spatial learning.

It is another puzzling factor why the short-run agents should be induced/incentivized or interested in clearly communicating its play's outcome, because as specified that the monopolist is not going to play with him anymore in the future, so the short-run entrant has no incentive in clearly communicating the incumbent's type. One of the ways to account for this is to assume that the outcome payoff later is shared between the agents or the presence of some utility transfer models. Another possibility is to introduce a hidden stage of communication coordination game, where the stochastically stable outcome is the state where every node chooses to tell the truth. This can be guaranteed by constructing this coordination game in a way that telling the truth and cooperating is the risk-dominant action (Goyal, Theorem 4.7 (2007) [12]).

If the players can also decide on the amount of effort to exert in information transmission, one has to introduce costs of transmissions and equal it to the marginal benefit from the long run cooperation in the repeated entry deterrence game. The differences in incentives to tell the truth will open a way for further research to the notion of maximally independent sets, which will in turn translate to properties of graphs and subgraphs, as well as the problem of experts and free riders in the form of central and peripheral players with specific degree connectivity and centrality measures.

A good avenue for future research can involve the analysis of understanding how these connections in the network can attribute or affect the welfare of entrants collectively and the incumbent in the long run. Connections in fact, and networks in general serve to perpetuate the inequality in complex systems since the distribution of degrees and connections are different among the nodes. These lead to differences in communication efforts and outcomes, which can affect individual payoffs and the networks' welfare in general.

In the sharing with costly information game, Goyal proves that adding new links can have conflicting effects on the set of equilibria and the payoffs of players, including the general payoff on network [12]. For example, adding links between maximally independent nodes leaves the equilibrium set the same, however otherwise it alters the equilibrium possibilities and leaves room for Pareto efficiency. This will not be a problem for our case, because of several useful properties of the model in this research. First, the belief functions are continuous and monotonic in the initial stages of the game. Adding a link cannot be harmful for any of the nodes as the communication can be made available to farther away nodes faster. Adding a link always creates a first order stochastically dominant geodesic distance distribution. Of course, useful properties also arise in different scenarios as well. For example, individual behavior in linear quadratic games is completely characterized by Bonacich centrality of players (Goyal (2007) [12]).

Note that one can argue that the transmission of information in this model relies on the fact that the agents have complete information about the structure of the network fully. In reality, agents only know the set of their neighbors and have minimal to no information about the full structure of the underlying network. The model at hand addresses this issue, because the transmission of play outcomes can be fully broken down into discrete interactions between players who are neighbors. For example, for a 4-step shortest path communication between player i and j that goes through players l, m, and n, it can be argued that i has no idea about its path to j. This path consists of sets $\{j, n\}$, $\{n, m\}$, $\{m, l\}$ and $\{l, i\}$ with nodes in each sets being neighbors and hence possessing complete information on each other without the need to have complete information about the structure of the whole graph.

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Appendices

Theorem .1. ***Growth Result, Székely (*1997*) [21]:*** *Let $G = (n, e)$ be a random graph. Let $\Psi_x(n)$ be the set of pairs with euclidean distance equal to x for any x. Then $\#\Psi_x(n) \precsim n^{\frac{4}{3}}$ for any x.*

Proof: First time that this result was established was by Spencer, Szemerédi and Trotter (1984) [19]. The proof idea shown here with all listed cases and possibilities is by Székely (1997) [21]. I will use the notation of order of magnitude throughout this analysis. A key result used in the proof is Szemerédi–Trotter theorem, which counts the number of point-line incidences. The idea of the proof is to deform any arbitrary graph to a simple graph with no loops or multiple edges, on which we can apply the Szemerédi–Trotter bound [21]. Fix an arbitrary distance x and denote $\#\Psi_x(n)$ the number of times this distance occurs between any pairs. From arbitrary agent i the agents who are positioned of euclidean distance x are located on circle of radius x with centre i. There is a bijection between the aim of this theorem and counting the number of point-circle incidences, but to be able to use Szemerédi–Trotter bound we need to make sure circles are mapped to lines, resulting in a simple graph. Thus, we will begin with point-circle graph C (nodes being the agents, edges being the arcs connecting nodes) and proceed by deleting edges to get a simple graph C', which will not be much different from C, so its properties can be transferred to the initial C. C and C' both have n nodes, but only C has $\#\Psi_x(n)$ edges. We proceed as follows: *Possibility* 1: There exists a loop in C, because only one point is of distance x from some agent i. We delete such circles, which are at most n. *Possibility* 2: Two points are on the same circle, so we delete multiple edges and at most $2n$ edges are deleted in this case. *Possibility* 3: Two points are on two distinct circles, so we delete at least one arc, which means the number of edges in C is at most halved. All three possibilities are illustrated in the Figure 4.

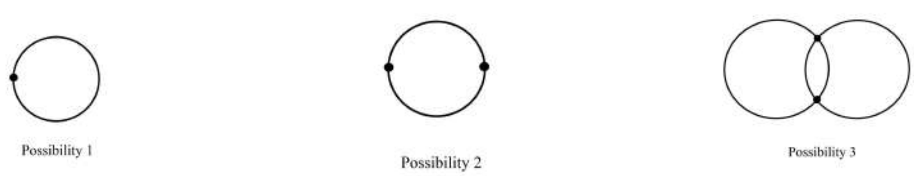

Figure 9

After this process, we get the simple graph C' which has at least $e' \geq \#\Psi_x(n) - \frac{1}{2}\#\Psi_x(n) - 3n = \frac{1}{2}\#\Psi_x(n) - 3n$ edges. We can now use the crossing number inequality and proceed with the exact cases in the standard proof of Szemerédi–Trotter theorem [20]. *Case* 1: $\frac{1}{2}\#\Psi_x(n) - 3n > 7n \implies$ we can apply the crossing number inequality $Cr(C') \gtrsim \frac{e'^3}{n^2}$. Given each crossing consists of 2 edges, its number should be less than the number of all the possible pairs of nodes, i. e. $\frac{e'^3}{n^2} \lesssim Cr(C') \lesssim \binom{n}{2} \lesssim n^2 \implies e'^3 \lesssim n^4 \implies e' \lesssim n^{\frac{4}{3}}$. Now, we showed that $e' \geq \frac{1}{2}\#\Psi_x(n) - 3n$. If $3n \leq \frac{1}{4}\#\Psi_x(n) \implies e' \geq \frac{1}{4}\#\Psi_x(n) \implies \frac{1}{4}\#\Psi_x(n) \leq e' \lesssim n^{\frac{4}{3}} \implies \#\Psi_x(n) \lesssim n^{\frac{4}{3}}$. If $3n \geq \frac{1}{4}\#\Psi_x(n)$, then $\#\Psi_x(n) \leq 12n$, hence $\#\Psi_x(n) \lesssim n \lesssim n^{\frac{4}{3}}$. *Case* 2: $\frac{1}{2}\#\Psi_x(n) - 3n < 7n \implies \#\Psi_x(n) \leq 20n$, hence $\#\Psi_x(n) \lesssim n \lesssim n^{\frac{4}{3}}$ for all cases. $\square$